RECITATION 9

APPLICATIONS OF OPTIMIZATION

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Section 1. *Exercises*

Exercise 1

Evaluate the following.

a. $\lim_{x\to\infty} e^{-x}$.
b. $\lim_{x\to-\infty} e^{x}$.
c. $\lim_{x\to\infty} x/e^{x}$.
d. $\lim_{x\to\infty}(\sin x)/x$.

Solution .:.

a. $\lim_{x\to\infty} e^{-x} = 0$ since e^x increases without bound. More formally, $e^{-x} = 1/e^x$. Since $e^x > x$ for $x > 0$, $0 \le e^{-x} \le \frac{1}{x}$. By the squeeze theorem, we get the result.
b. $\lim_{x\to-\infty} e^{x} = \lim_{x\to\infty} e^{-x} = 0$.
c. $\lim_{x\to\infty} x/e^{x} = 0$ since $e^x > x^2$ for all $x > 0$.
d. $|\sin(x)| \le 1$ so that $-1/x \le (\sin x)/x \le 1/x$. Since both $\lim_{x\to\infty} 1/x = \lim_{x\to\infty} -1/x = 0$, the squeeze theorem yields that $0 \le \lim_{x\to\infty}(\sin x)/x \le 0$ and thus we have equality.

Exercise 2

What are the vertical asymptotes of $\tan(1/x)$?

Solution .:.

Note that $\tan(1/x) = \sin(1/x)/\cos(1/x)$. When $\cos(1/x) = 0$, $\sin(1/x) = \pm 1$. As continuous functions, $\tan(1/x)$ has a vertical asymptote whenever $\cos(1/x) = 0$. This happens whenever $1/x = \pi/2 + \pi n$ for some integer n. Therefore $x = \frac{2}{\pi(2n+1)}$ is a vertical asymptote for each integer n. This means that $\tan(1/x)$ has infinitely many vertical asymptotes.

Exercise 3

What are the horizontal asymptotes of $f(x) = \frac{x-3}{3x+1}$?
What are the vertical asymptotes of f?

Solution .:.

We need to consider the two limits $\lim_{x\to\pm\infty} f(x)$. FIrstly, note that we can rewrite

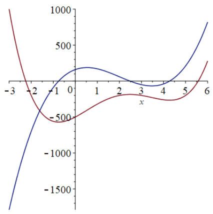
$$f(x) = \frac{x-3}{3x+1} = \frac{1-\frac{3}{x}}{3+\frac{1}{x}}.$$

So when we take the limit, the limit of the numberator is $1+0 = 1$ and the limit of the denominator is $3+0 = 3$. Hence the limit of the quotient $\lim_{x\to\pm\infty} f(x) = 1/3$. Therefore $y = 1/3$ is the only horizontal asymptote.

The vertical asymptotes of f occur when the denominator is 0 and the numerator isn't 0. The denominator is 0 iff $3x+1 = 0$ iff $x = -1/3$. When this occurs, $x - 3 \neq 0$, and therefore $x = -1/3$ is the only vertical asymptote.

Exercise 4

Identify which curve is the derivative of the other:



Solution .:.

If f is the derivative of g, then when f is below 0, g should be decreasing. Note that the red function is below 0 for most of the pictured graph, but the blue function increases at times. Hence red cannot be the derivative of blue, and thus blue must be the derivative of red.

Exercise 5

Where is $f(x) = 10x^3 + 2x^2 + 5x + 6$ concave up? Concave down? What are the inflection points of f?

Solution .:.

$f'(x) = 30x^2 + 4x + 5$. $f''(x) = 60x + 4$. This is less than 0 iff $60x + 4 < 0$ iff $60x < -4$ iff $x < -1/15$. So f is concave down on $(-\infty, -1/15)$.

$f''(x) > 0$ iff $60x + 4 > 0$ iff $x > -1/15$. Hence f is concave up on $(1/15, \infty)$.

Therefore $x = 1/15$ yields an inflection point of $(1/15, f(1/15))$.

Exercise 6

Calculate the relative extrema of $f(x) = e^x - x$ above using the first derivative test. Calculate the relative extrema of f using the second derivative test.

Solution .:.

$f'(x) = e^x - 1$. This is always defined and is 0 iff $e^x = 1$, iff $x = 0$. Thus $x = 0$ is the only critical point.

For the first derivative test, we need to check the sign of $f'(x)$ when $x < 0$ and when $x > 0$. For $x < 0$, $e^x < 1$ and therefore $f'(x) < 0$. For $x > 0$, $e^x > 1$ and therefore $f'(x) > 0$. Therefore $x = 0$ yields a relative minimum.

$f''(x) = e^x$. Since $f''(0) = e^0 = 1$ is positive, it follows that $x = 0$ gives a relative minimum (f is concave up around $x = 0$).

Exercise 7

How many horizontal asymptotes can a function have? What are the horizontal asymptotes of $f(x) = \frac{|x|+1}{x+1}$?

Solution .:.

A function can have at most two horizontal asymptotes: one for each limit $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Hence 0, 1, and 2 are the only possible numbers of horizontal asymptotes of a function.

$f(x)$ as defined here has two (the maximum number) of horizontal asymptotes: for $x < 0$, $|x| = -x$ and therefore

$$f(x) = \frac{-x+1}{x+1} = \frac{-1+1/x}{1+1/x},$$

where then $\lim_{x\to-\infty} f(x) = \frac{-1+0}{1+0} = -1$. For $x > 0$, $|x| = x$ and therefore $f(x) = \frac{x+1}{x+1} = 1$ so that $\lim_{x\to\infty} f(x) = 1$.